1. MATHEMATICAL INDUCTION

<u>EXAMPLE 1</u>: Prove that

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \tag{1.1}$$

for any integer $n \geq 1$.

<u>Proof</u>:

STEP 1: For n=1 (1.1) is true, since

$$1 = \frac{1(1+1)}{2}.$$

STEP 2: Suppose (1.1) is true for some $n = k \geq 1$, that is

$$1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}.$$

STEP 3: Prove that (1.1) is true for $n = k + 1$, that is

$$1 + 2 + 3 + \ldots + k + (k+1) \stackrel{?}{=} \frac{(k+1)(k+2)}{2}.$$

We have

$$1 + 2 + 3 + \ldots + k + (k+1) \stackrel{\textbf{ST.2}}{=} \frac{k(k+1)}{2} + (k+1) = (k+1)\left(\frac{k}{2} + 1\right) = \frac{(k+1)(k+2)}{2}. \blacksquare$$

<u>EXAMPLE 2</u>: Prove that

$$1 + 3 + 5 + \ldots + (2n-1) = n^2 \tag{1.2}$$

for any integer $n \geq 1$.

<u>Proof</u>:

STEP 1: For n=1 (1.2) is true, since $1 = 1^2$.

STEP 2: Suppose (1.2) is true for some $n = k \geq 1$, that is

$$1 + 3 + 5 + \ldots + (2k-1) = k^2.$$

STEP 3: Prove that (1.2) is true for $n = k + 1$, that is

$$1 + 3 + 5 + \ldots + (2k-1) + (2k+1) \stackrel{?}{=} (k+1)^2.$$

We have: $1 + 3 + 5 + \ldots + (2k-1) + (2k+1) \stackrel{\textbf{ST.2}}{=} k^2 + (2k+1) = (k+1)^2.$ $\blacksquare$

EXAMPLE 3: Prove that

$$n! \leq n^n \tag{1.3}$$

for any integer $n \geq 1$.

Proof:

STEP 1: For n=1 (1.3) is true, since $1! = 1^1$.

STEP 2: Suppose (1.3) is true for some $n = k \geq 1$, that is $k! \leq k^k$.

STEP 3: Prove that (1.3) is true for $n = k+1$, that is $(k+1)! \overset{?}{\leq} (k+1)^{k+1}$. We have

$$(k+1)! = k! \cdot (k+1) \overset{\textbf{ST.2}}{\leq} k^k \cdot (k+1) < (k+1)^k \cdot (k+1) = (k+1)^{k+1}. \blacksquare$$

EXAMPLE 4: Prove that

$$8 \mid 3^{2n} - 1 \tag{1.4}$$

for any integer $n \geq 0$.

Proof:

STEP 1: For n=0 (1.4) is true, since $8 \mid 3^0 - 1$.

STEP 2: Suppose (1.4) is true for some $n = k \geq 0$, that is $8 \mid 3^{2k} - 1$.

STEP 3: Prove that (1.4) is true for $n = k+1$, that is $8 \mid 3^{2(k+1)} - 1$. We have

$$3^{2(k+1)} - 1 = 3^{2k+2} - 1 = 3^{2k} \cdot 9 - 1 = 3^{2k}(8+1) - 1 = \underbrace{3^{2k} \cdot 8}_{\text{div. by } 8} + \underbrace{3^{2k} - 1}_{\substack{\text{St. 2} \\ \text{div. by } 8}}. \blacksquare$$

EXAMPLE 5: Prove that

$$7 \mid n^7 - n \tag{1.5}$$

for any integer $n \geq 1$.

Proof:

STEP 1: For n=1 (1.5) is true, since $7 \mid 1^7 - 1$.

STEP 2: Suppose (1.5) is true for some $n = k \geq 1$, that is

$$7 \mid k^7 - k.$$

STEP 3: Prove that (1.5) is true for $n = k+1$, that is $7 \mid (k+1)^7 - (k+1)$. We have

$$\begin{aligned}(k+1)^7 - (k+1) &= k^7 + 7k^6 + 21k^5 + 35k^4 + 35k^3 + 21k^2 + 7k + 1 - k - 1 \\ &= \underbrace{k^7 - k}_{\substack{\text{St. 2} \\ \text{div. by } 7}} + \underbrace{7k^6 + 21k^5 + 35k^4 + 35k^3 + 21k^2 + 7k}_{\text{div. by } 7}. \blacksquare\end{aligned}$$